Theory of Finite-Range Distorted-Waves Calculations*

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Most distorted-waves direct-reaction calculations rely upon a "zero-range approximation." In the present article some of the errors caused by this approximation are discussed, and a general method suitable for numerical computation is described which does not use this approximation. Applications to stripping and knock-on reactions are described in detail.

I. INTRODUCTION

THE distorted-waves Born approximation (DWB) is used in nearly all nuclear direct-reaction theories.[1,2] In this approximation the transition amplitude is computed as a first-order matrix element between channel wave functions for the colliding systems A, a and the separating systems B, b. That is, the DWB transition amplitude for the reaction $A(a,b)B$ has the form of a matrix element between product wave functions:

$$T_{ba}=(\psi_B\psi_b\chi_b^{(-)}(\mathbf{k}_b,\mathbf{r}_b),V\psi_A\psi_a\chi_a^{(+)}(\mathbf{k}_a,\mathbf{r}_a)). \quad (1)$$

Here ψ_B, ψ_b, ψ_A, ψ_a are the internal wave functions for the noninteracting, separated particles B, b, A, a. The interaction V is the interaction whose off-diagonal matrix elements are responsible for the transition, and its precise meaning depends on the particular reaction mechanism being studied. The functions $\chi_b^{(-)}$ and $\chi_a^{(+)}$ are the "distorted waves."[1,2] They are elastic scattering wave functions which describe the relative motion of the pair A, a before the collision, or of the pair B, b after the collision. Here $\chi_a^{(+)}$ is a function of $\mathbf{r}_a$, the displacement of a from A, and $\chi_b^{(-)}$ is a function of $\mathbf{r}_b$, the displacement of b from B. In practice, the functions $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are generated from optical model potentials which are chosen to give a fit to the observed elastic scattering in the entrance and exit channels.

Matrix elements of the form of Eq. (1) describe many kinds of processes, such as inelastic scattering, deuteron stripping, many-particle stripping, etc. Both the direct and exchange terms for these processes are of the form of Eq. (1). (Detailed discussions of some individual processes are given in later sections.)

Evidently it is important to have accurate numerical evaluations of Eq. (1) in order to have useful comparisons of the DWB theory with experiment. Unfortunately the numerical evaluation of Eq. (1) is difficult, so that simplifying approximations usually are introduced. The so-called "zero-range approximation" is especially important, and is used almost always in distorted-waves calculations which concern rearrangement collisions. This approximation owes its importance to the fact that in a rearrangement process no two of the wave functions which appear in Eq. (1) have quite the same argument. In particular, the vectors $\mathbf{r}_a$ and $\mathbf{r}_b$ differ from each other. Thus, the evaluation of T_{ba} requires at least a six-dimensional integration, over the space of these two vectors. The other arguments are related linearly to $\mathbf{r}_a$ and $\mathbf{r}_b$. However, when these relations are used to express the integrand in terms of a nonredundant set of independent variables there is not, in general, any analytic simplification. Thus, T_{ba} has to be computed numerically as a general integral in six dimensions, whereas, the zero-range approximation introduces a delta function in the integrand which reduces T_{ba} to a three-dimensional integral.

To understand better the zero-range approximation, it is convenient to first isolate in Eq. (1) the matrix element of the interaction V, taken between the internal states:

$$\langle Bb|V|Aa\rangle=\int\psi_B^*\psi_b^*V\psi_A\psi_a d\xi. \quad (2)$$

Here ξ represents all the coordinates independent of $\mathbf{r}_a$ and $\mathbf{r}_b$. Thus Eq. (2) expresses those portions of the calculation of T_{ba} which involve the internal states, and which do not concern the scattering wave functions $\chi_a^{(+)}$ and $\chi_b^{(-)}$. As a result, the calculations of Eq. (2) generally are fairly easy, and largely analytic. (Also it is helpful for physical understanding that calculations involving the internal states be separated from those involving the scattering states, because these are understood on different terms.) Of course the matrix element (2) remains a function of $\mathbf{r}_a$ and $\mathbf{r}_b$. It plays the role of an effective interaction or form factor for the transition between the elastic scattering states $\chi_a^{(+)}$ and $\chi_b^{(-)}$. It

* A preliminary account of this work has been given elsewhere. [E. C. Halbert, R. M. Drisko, G. R. Satchler, and N. Austern, in *Proceedings of Rutherford Jubilee Conference, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1962)].

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[1] N. Austern, in *Fast Neutron Physics, II*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1963).

[2] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, New York, 1961).

contains all the information on nuclear structure, angular momentum selection rules, and even the type of reaction being considered (whether stripping, or inelastic scattering, or knock-on, etc.). In terms of this matrix element the calculation of the DWB amplitude then is completed in the form

$$T_{ba}=\int d\mathbf{r}_a \int d\mathbf{r}_b \chi_b^{(-)*}(\mathbf{k}_b,\mathbf{r}_b) \times\langle Bb|\,V\,|Aa\rangle\chi_a^{(+)}(\mathbf{k}_a,\mathbf{r}_a). \quad (3)$$

It is Eq. (3) which presents problems of calculation. The integration still extends over both the variables $\mathbf{r}_a$ and $\mathbf{r}_b$. Both these variables appear in the form factor, in a way that generally makes Eq. (3) a difficult six-dimensional numerical integral. In most current distorted wave calculations, this difficulty is removed by the assumption that the form factor (2) has a very small range in $(\mathbf{r}_{aA}-\mathbf{r}_{bA})$, perhaps because V has a small range, or perhaps because the internal wave functions have small ranges. This zero-range assumption has the physical meaning that particle b is assumed to be emitted at the same point at which particle a is absorbed, so that $\mathbf{r}_b=(M_A/M_B)\mathbf{r}_a$, where M_A and M_B are the masses of A and B. As a result, Eq. (3) is reduced to a three-dimensional integral, and this greatly facilitates its computation.

Only for the direct term of inelastic scattering (and with a local interaction) is the zero-range approximation not used. In this simple case the variables $\mathbf{r}_b$ and $\mathbf{r}_a$ are equal, the form factor becomes a function of only one variable, and the matrix element (3) automatically becomes a three-dimensional integral.

It has not been clear up to the present how much the zero-range approximation affects the calculated angular distributions, and in what cases the effects are strongest, etc. For some guide to these effects one can consider the Fourier transforms of the product $\chi_a^{(+)}\chi_b^{(-)*}$ and of the form factor (2), using as one of the two variables in these expansions the momentum conjugate to $\mathbf{r}_b-(M_A/M_B)\mathbf{r}_a$ (the argument of the delta function in the zero-range approximation). Then the zero-range approximation is based on the assumption that the important components of this momentum in $\chi_b^{(-)*}\chi_a^{(+)}$ are much smaller than the momenta at which the Fourier transform of the form factor (2) begins to decrease appreciably. This implies the approximation will be most accurate in reactions which have small momentum transfer, such as medium-energy deuteron stripping, or inelastic scattering. It is expected to be inaccurate for exchange terms, or for "heavy-particle stripping" terms, because in these cases the form factor is not very localized, and itself involves small momenta. The zero-range approximation probably also is rather inaccurate in treating the contributions from reactions occurring inside the nucleus; it probably over emphasizes the importance of such contributions. The average momenta in $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are greatest in the nuclear interior, so that there, especially, the delta function of the zero-range approximation introduces correlations between these rapidly varying functions, without which considerable cancellations might occur in the integral (3). So there has been much interest in eliminating this zero-range approximation from distorted wave calculations,[3] and in understanding the specific effects which are related to its elimination.

In Sec. 2 we give some further qualitative discussion of finite-range effects. Then in Sec. 3 we describe a scheme of numerical calculation by which Eq. (3) is computed exactly. Later sections give specific formulas for the cases of stripping and exchange scattering, while the treatment of a general form factor is indicated in the Appendix. Detailed numerical results of the application of this theory to various nuclear reactions will be described in a later publication.

The procedure to be described makes possible the accurate evaluation of conventional DWB matrix elements, all of which are of the form of Eq. (3). However, there are other aspects of the conventional treatment which may be questioned, aside from the zero-range approximation itself. For example,

(i) weak coupling is assumed; that is, the theory is first order in the interaction V, and higher orders are assumed to be negligible.

(ii) even within the nucleus, it is assumed that the motion of a complex particle such as a deuteron may be described in terms of its center-of-mass motion, without dissociation or internal excitation of itself or the nucleus (except insofar as this can be described by simple absorption).

(iii) usually, certain interaction terms, such as that between the proton and target nucleus in a (d,p) reaction, or some exchange terms, such as heavy particle stripping, are neglected.

The question may be raised whether it makes sense to correct the zero-range approximation without simultaneously—or first—correcting some of the others. In particular, the finite-range correction involves detailed correlations within the nuclear interior, the very region where the approximate wave functions are at their worst.

Nevertheless, we believe it reasonable and worthwhile to determine the effects of finite range superimposed on a conventional distorted-wave treatment. The zero-range approximation is probably by far the easiest one to correct. Also qualitative results may be enlightening. First: there may be some interesting changes in the nuclear surface region, where the approximate wave functions are probably quite good. Second: just because contributions to the reaction integral (3) from the nuclear interior are particularly subject to error, it is of interest to determine whether their importance depends on the zero-range approxi-

[3] E. C. Halbert, R. M. Drisko, G. R. Satchler, and N. Austern, in *Proceedings of Rutherford Jubilee Conference, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1962).

mation. If the introduction of finite range does produce a general suppression of these contributions, then errors in the interior wave functions become less disturbing. Finally, concerning point (iii) above, a finite-range computer program will for the first time allow a realistic appraisal of the importance of many of these terms.

II. QUALITATIVE DISCUSSION OF FINITE-RANGE EFFECTS: PLANE WAVES AND LOCAL WKB

An exact treatment of finite-range effects is easy in the *plane-wave* Born approximation, and the results obtained in this manner are familiar in the important case of deuteron stripping.[1,2,4] It is interesting to review these results, and to indicate qualitatively how they may extend to other reactions and in what way they may change if distorted waves are used.

The coordinate system for a (d,p) reaction is shown in Fig. 1, with $n=x$, $d=a$, $p=b$. In terms of the general notation of the preceding section, the variables indicated here have the meanings: $\mathbf{r}_a=\mathbf{r}_d$, $\mathbf{r}_b=\mathbf{r}_p$, while $\mathbf{r}_{xA}=\mathbf{r}_n$. If for the present discussion we ignore the spins of the neutron and proton and the target nucleus, then $\psi_b=1$ and $\psi_A=1$, and the transition amplitude reduces to the form

$$T_{dp}=\int d\mathbf{r}_p\int d\mathbf{r}_n\chi_p^{(-)*}(\mathbf{k}_p,\mathbf{r}_p)\psi_n^*(\mathbf{r}_n)\times V(|\mathbf{r}_n-\mathbf{r}_p|)\psi_d(|\mathbf{r}_n-\mathbf{r}_p|)\chi_d^{(+)}(\mathbf{k}_d,\mathbf{r}_d). \quad (4)$$

Here ψ_n is the wave function for the captured neutron. It is convenient to introduce the abbreviation

$$D(r_{pn})=V(r_{pn})\psi_d(r_{pn}), \quad (5)$$

where $\mathbf{r}_{pn}=\mathbf{r}_p-\mathbf{r}_n$. If the interaction V has zero range this quantity D reduces to a delta function:

$$D=V\psi_d=-(4\pi)^{1/2}(\hbar^2/M)N\delta(\mathbf{r}_{pn}) \quad \text{(zero-range)}, \quad (6)$$

where M is the nucleon mass and N is the asymptotic normalization factor for the deuteron. From effective-range theory,

$$N=(2\gamma)^{1/2}[1-\gamma\rho_t(-\epsilon,-\epsilon)]^{-1/2},$$

where $\epsilon=\hbar^2\gamma^2/M$ is the deuteron binding energy, and $\rho_t(-\epsilon,-\epsilon)$ is the triplet effective range.[1] If the interaction has finite range then Eq. (6) is not valid, but will be useful for comparison. In either case the amplitude (4) is

$$T_{dp}=\int d\mathbf{r}_p\int d\mathbf{r}_n\chi_p^{(-)*}(\mathbf{k}_p,\mathbf{r}_p)\times[\psi_n^*(\mathbf{r}_n)D(r_{pn})]\chi_d^{(+)}(\mathbf{k}_d,\mathbf{r}_d). \quad (7)$$

The quantity in brackets is the "form factor," discussed in the Introduction. In zero range it reduces to a delta function times the bound-neutron wave function, and the integral becomes a three-dimensional integral.

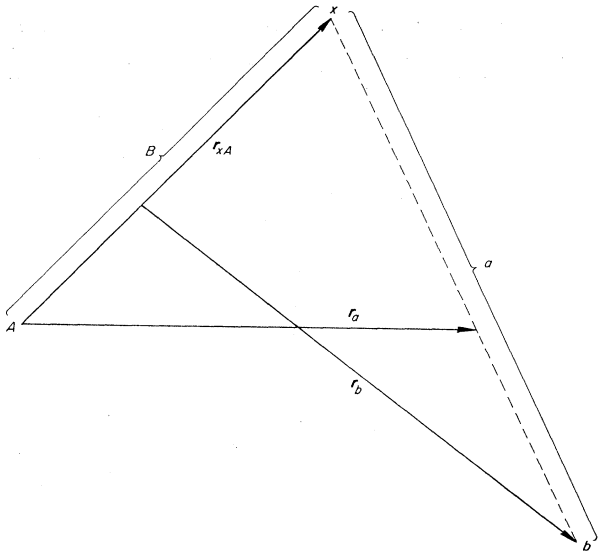

FIG. 1. The coordinate vectors for a general stripping reaction.

In plane-wave Born approximation Eq. (7) can be factored, even with finite range. First, the use of geometrical relations among the variables converts this equation to the form

$$T_{dp}=\int d\mathbf{r}_n\int d\mathbf{r}_{pn}\chi_p^{(-)*}(\mathbf{k}_p,(M_A/M_B)\mathbf{r}_n+\mathbf{r}_{pn})\times[\psi_n^*(\mathbf{r}_n)D(r_{pn})]\chi_d^{(+)}(\mathbf{k}_d,\mathbf{r}_n+\tfrac{1}{2}\mathbf{r}_{pn}). \quad (8)$$

Then replacement of the distorted waves by plane waves yields

$$T_{dp}=\int\psi_n^*(\mathbf{r}_n)\exp i[\mathbf{r}_n\cdot(\mathbf{k}_d-(M_A/M_B)\mathbf{k}_p)]d\mathbf{r}_n\times\int D(r_{pn})\exp i[\mathbf{r}_{pn}\cdot(\tfrac{1}{2}\mathbf{k}_d-\mathbf{k}_p)]d\mathbf{r}_{pn}. \quad (9)$$

Because the arguments of the χ_p and χ_d in Eq. (8) are each displaced from the common value $\mathbf{r}_n$, there appears in Eq. (9) an averaging of the finite-range function D. This averaging involves the momenta of the continuum wave functions. Finite-range effects express themselves in Eq. (9) as a departure of the second factor from the simple constant value $-(4\pi)^{1/2}(\hbar^2/M)N$, which it assumes in zero range. For example, if a Hulthén wave function is used for the deuteron, the second integral of Eq. (9) becomes

$$-(4\pi)^{1/2}(\hbar^2/M)N\{(\zeta^2-\gamma^2)[\zeta^2+(\tfrac{1}{2}\mathbf{k}_d-\mathbf{k}_p)^2]^{-1}\}, \quad (10)$$

where $(\zeta/\gamma)\approx 7$, as usual. Eq. (10) is familiar from discussions of Butler stripping theory.[1,2,4] Experience with the application of Eq. (10) for medium-energy stripping reactions has shown that the finite-range cor-

[4] S. T. Butler, *Nuclear Stripping Reactions*, (John Wiley & Sons, Inc., New York, 1957).

rection factor usually introduces only minor changes in the shape of the Butler angular distribution, but reduces the magnitude of the peak cross sections by about 20%.[5]

Thus, there is a well-marked tendency for finite-range effects to reduce the cross section, because of averaging of the variable $\mathbf{r}_{pn}$ over the oscillations of the continuum wave functions. Inspection of Eq. (8) suggests that a similar averaging should occur if distorted waves are used, but with the modification that in the finite-range correction factor the asymptotic momenta $\mathbf{k}_p$ and $\mathbf{k}_d$ should be replaced by the "local momenta" of the waves $\chi_p^{(-)}$ and $\chi_d^{(+)}$ in the vicinity of the common argument $\mathbf{r}_n$. In a formal approach these local momenta are found in the "local WKB approximation," in which trigonometric interpolation is used to represent the distorted waves $\chi_p^{(-)}$ and $\chi_d^{(+)}$ within a small region about each point $\mathbf{r}_n$. For example, in Eq. (8) the function $\chi_d^{(+)}$ depends on the two variables $\mathbf{r}_n$ and $\mathbf{r}_{pn}$. In the local WKB method we approximate the dependence on $\mathbf{r}_{pn}$ as being trigonometric, with a wavelength determined from the local kinetic energy, and with an amplitude and phase determined from the properties of $\chi_d^{(+)}(\mathbf{k}_d,\mathbf{r}_n)$. However, in whichever way the local momenta are introduced, it is evident that we thereby obtain a finite-range correction factor which depends on $\mathbf{r}_n$, which must then be carried in the integration over $\mathbf{r}_n$. (At present some use is being made[6] of the local WKB method in numerical calculations.) We can next argue that because momenta are expected to be at their largest in the nuclear interior, the most important finite-range averaging is expected to occur there, with a consequent reduction of the relative importance of the contributions to the reaction amplitude from the nuclear interior. Indeed it is very appealing to find a reason for anticipating some suppression of contributions from the nuclear interior, inasmuch as it is already known that such suppression often improves the agreement between zero-range DWB calculations and experiment.[7]

In the finite-range correction factor of Eq. (9), the momenta $\mathbf{k}_p$ and $\mathbf{k}_d$ are subtracted. Hence, at the important forward scattering angles in the (d,p) reaction, the averaging of D involves a net momentum $(\mathbf{k}_p-\frac{1}{2}\mathbf{k}_d)$ which, at the medium energies usually employed, is not very large compared to the inverse of the range of $D(r_{pn})$ (which is of the order 1.5 F). However, in other types of reactions, the averaging may involve net momenta and ranges which are much larger, and may cause major modifications of the cross sections. Such may be the case for so-called heavy-particle stripping; for a (d,p) reaction the appropriate momentum is then $(\mathbf{k}_p+\mathbf{k}_d/M_A)$ and the "range" is of the order of the nuclear radius. Knock-on scattering, discussed in Sec. VI, also gives rise to large momentum transfers, and is accordingly sensitive to finite-range effects.

These qualitative ideas are discussed again in Sec. VI. It will be noted there that the above discussion of them has been somewhat over-simplified.

III. DERIVATION OF THE EXACT FORMULAS

We now describe the procedures by which the distorted-waves amplitude of Eq. (3) is computed numerically, without use of the zero-range approximation. It will be seen that the basic formal structure of the calculation, in its use of spherical harmonic expansions, is the same as in the zero-range case, so that much of the coding done for zero-range calculations can be taken over intact, and so that detailed comparison with zero-range results is straightforward.

The amplitude (3) must be specified more carefully. If the spins of the particles are designated J_A, J_B, s_a, and s_b, and their corresponding z components by M_A, M_B, m_a, and m_b, respectively, then the amplitude (1) becomes

$$T_{ba}\equiv\langle J_BM_B,s_bm_b,\mathbf{k}_b|\,V\,|J_AM_A,s_am_a,\mathbf{k}_a\rangle$$
$$=\int d\mathbf{r}_a\int d\mathbf{r}_b\chi_b^{(-)*}(\mathbf{k}_b,\mathbf{r}_b)$$
$$\times\langle J_BM_B,s_bm_b|\,V\,|J_AM_A,s_am_a\rangle\chi_a^{(+)}(\mathbf{k}_a,\mathbf{r}_a). \quad (11)$$

It is convenient to expand the form-factor matrix element in the usual manner,[8,9] such that each term in the expansion corresponds to the transfer to the target nucleus of a definite angular momentum j, which in turn is comprised of an orbital part l and a spin part s, according to the vector coupling

$$\mathbf{j}=\mathbf{J}_B-\mathbf{J}_A,\quad \mathbf{s}=\mathbf{s}_a-\mathbf{s}_b,\quad \mathbf{j}=\mathbf{l}+\mathbf{s}.$$

The multipole series for the matrix element can be written with Clebsch-Gordan coefficients[10]

$$\langle J_BM_B,s_bm_b|\,V\,|J_AM_A,s_am_a\rangle$$
$$=\sum_{lsj}\langle J_AjM_AM_B-M_A|\,J_BM_B\rangle$$
$$\times\langle lsmm_a-m_b|\,jM_B-M_A\rangle$$
$$\times\langle s_as_bm_a,\,-m_b|\,sm_a-m_b\rangle(-)^{s_b-m_b}i^{-l}$$
$$\times G_{lsjm}(\mathbf{r}_b,\mathbf{r}_a;\,bB,aA), \quad (12)$$

where $m=M_B+m_b-M_A-m_a$. The symbols bB, aA

[5] A. Hamburger (private communication); C. R. Lubitz, *Numerical Table of Butler-Born Approximation Stripping Cross Sections* (University of Michigan, Ann Arbor, Michigan, 1957).
[6] W. R. Gibbs (private communication).
[7] R. H. Bassel, R. M. Drisko and G. R. Satchler, Bull. Am. Phys. Soc. 8, 57 (1963); G. R. Satchler, in *Direct Interactions and Nuclear Reaction Mechanisms*, edited by E. Clementel and C. Villi (Gordon and Breach, Science Publishers, New York, 1963).
[8] G. R. Satchler, Nucl. Phys. 18, 110 (1960).
[9] R. H. Bassel, R. M. Drisko and G. R. Satchler, Oak Ridge National Laboratory Report No. 3240 (unpublished).
[10] D. M. Brink and G. R. Satchler, *Angular Momentum* (Oxford University Press, New York, 1962).

as arguments of G denote the dependence on the various nuclear quantum numbers. It will be convenient later to have the inverted form of this expansion

$$G_{lsjm}=i^l[(2l+1)/(2J_B+1)]$$
$$\times\sum_{M_BM_Am_bm_a}(-)^{s_b-m_b}\langle lsmm_a-m_b|jM_B-M_A\rangle$$
$$\times\langle J_AjM_AM_B-M_A|J_BM_B\rangle\langle s_as_bm_a,-m_b|sm_a-m_b\rangle$$
$$\times\langle J_BM_B,s_bm_b|V|J_AM_A,s_am_a\rangle. \quad (13)$$

The phase factor i^l is included to ensure convenient time reversal properties.[10] By its construction, G_{lsjm} transforms under a rotation of the coordinate system like the spherical harmonic $Y_l{}^{m*}$. Another property carried by G is the parity of the nuclear transition,

$$\Pi=\pi(a)\pi(A)\pi(b)\pi(B),$$

where $\pi(i)$ is the parity of the internal state of particle i. [Since G is a function of *two* position vectors, this parity is not generally related to l. Only in the zero-range approximation is the parity of G necessarily given by $(-)^l$.] It is helpful to write G as the product of two factors,

$$G_{lsjm}(\mathbf{r}_b,\mathbf{r}_a)=A_{lsj}f_{lsjm}(\mathbf{r}_b,\mathbf{r}_a). \quad (14)$$

The separation into a spectroscopic coefficient (or "reduced width") A_{lsj} and the form factor $f_{lsj,m}$ is one of convenience, so that for example, standard types of form factors with simple normalization may be used in computation. In general A_{lsj} carries such quantities as fractional parentage coefficients for the initial state and final state, and also the overlap integral of the two parent states.

The expansion (12) has the convenience that (in the absence of spin-orbit coupling in the distorted waves) the different values of l, s, and j contribute incoherently to the differential cross section. Even with spin-orbit coupling, j remains incoherent. In any case, often only one value of l, s, and j is allowed, or is important, and we shall concentrate on one such term. Neglecting spin-orbit coupling, we then define partial amplitudes by

$$(2l+1)^{\frac{1}{2}}i^l\beta_{sj}{}^{lm}(\mathbf{k}_b,\mathbf{k}_a)$$
$$\equiv\int d\mathbf{r}_a\int d\mathbf{r}_b\chi_b{}^{(-)*}(\mathbf{k}_b,\mathbf{r}_b)f_{lsjm}(\mathbf{r}_b,\mathbf{r}_a)\chi_a{}^{(+)}(\mathbf{k}_a,\mathbf{r}_a). \quad (15)$$

The differential cross section is then given by

$$\frac{d\sigma}{d\omega}=\frac{\mu_a\mu_b}{(2\pi\hbar^2)^2}\frac{k_b}{k_a}\frac{2J_B+1}{2J_A+1}\sum_{lsj}\frac{|A_{lsj}|^2}{2s_a+1}\sum_m|\beta_{sj}{}^{lm}|^2,$$

where μ_a is the reduced mass of the pair a, A, etc. (From now on, for simplicity we will drop the labels s and j, as these play no part in the calculation now to be described.) The finite-range problem has become the problem of evaluating Eq. (15).

For the distorted waves in Eq. (14) we use the familiar[1,2,9] partial-waves expansions

$$\chi_a{}^{(+)}(\mathbf{k}_a,\mathbf{r}_a)=(4\pi/k_ar_a)$$
$$\times\sum_{L_aM_a}i^{L_a}\chi_{L_a}{}^{(a)}(k_a,r_a)Y_{L_a}{}^{M_a}(\hat{r}_a)Y_{L_a}{}^{M_a*}(\hat{k}_a), \quad (16a)$$

$$\chi_b{}^{(-)*}(\mathbf{k}_b,\mathbf{r}_b)=(4\pi/k_br_b)$$
$$\times\sum_{L_bM_b}i^{-L_b}\chi_{L_b}{}^{(b)}(k_b,r_b)Y_{L_b}{}^{M_b}(\hat{r}_b)Y_{L_b}{}^{M_b*}(\hat{k}_b). \quad (16b)$$

Here $\hat{r}$ denotes the polar angles of the vector $\mathbf{r}$, etc. These expansions are almost the only orderly approach to computation of the distorted waves, and are integral parts of all accurate distorted-waves computer programs. Also, since the form factor f_{lm} transforms like $Y_l{}^{m*}$, its expansion into a double series in spherical harmonics of $\hat{r}_a$ and $\hat{r}_b$ takes the form

$$f_{lm}(\mathbf{r}_b,\mathbf{r}_a)$$
$$=\sum_{L_aL_bM}F_{lL_bL_a}(r_b,r_a)Y_{L_b}{}^{M*}(\hat{r}_b)Y_{L_a}{}^{m-M*}(\hat{r}_a)$$
$$\times\langle L_bL_aMm-M|lm\rangle. \quad (17)$$

The Clebsch-Gordan coefficient displays the selection rule

$$|L_a-L_b|\leqslant l\leqslant L_a+L_b,$$

which limits the double summation. By inversion we have

$$F_{lL_bL_a}(r_b,r_a)$$
$$=\sum_M\langle L_bL_aMm-M|lm\rangle$$
$$\times\int d\hat{r}_a\int d\hat{r}_bf_{lm}(\mathbf{r}_b,\mathbf{r}_a)Y_{L_b}{}^M(\hat{r}_b)Y_{L_a}{}^{m-M}(\hat{r}_a). \quad (18)$$

Then in terms of this expansion of the form factor and in terms of the partials wave-expansions (16), the result of integration of Eq. (15) becomes

$$(2l+1)^{\frac{1}{2}}\beta^{lm}(\mathbf{k}_b,\mathbf{k}_a)=(16\pi^2/k_bk_a)\sum_{L_aL_b}i^{L_a-L_b-l}I_{L_bL_a}{}^l$$
$$\times\sum_M\langle L_bL_aMm-M|lm\rangle Y_{L_b}{}^{M*}(\hat{k}_b)Y_{L_a}{}^{m-M*}(\hat{k}_a), \quad (19)$$

where the radial integrals are

$$I_{L_bL_a}{}^l=\int_0^\infty r_adr_a\int_0^\infty r_bdr_b\chi_{L_b}{}^{(b)}(k_b,r_b)$$
$$\times F_{lL_bL_a}(r_b,r_a)\chi_{L_a}{}^{(a)}(k_a,r_a). \quad (20)$$

The expression (19) simplifies if we choose the z axis along $\mathbf{k}_a$ and the y axis along $\mathbf{k}_a\times\mathbf{k}_b$, and put θ as the angle between $\mathbf{k}_a$ and $\mathbf{k}_b$; then for $m\geq 0$,

$$\beta^{lm}(\theta)=(4\pi/k_ak_b)$$
$$\times\sum_{L_aL_b}i^{L_a+L_b-l}(2L_b+1)^{\frac{1}{2}}\langle L_blm,-m|L_a0\rangle$$
$$\times[(L_b-m)!/(L_b+m)!]^{\frac{1}{2}}P_{L_b}{}^m(\theta)I_{L_bL_a}{}^l, \quad (21)$$

while for $m<0$ we have the symmetry relation

$$\beta^{lm}=\Pi(-)^m\beta^{l-m}.$$

Here Π is the parity of the nuclear transition, hence of the form factor, as discussed earlier. Thus, the finite-range problem reduces to the problem of evaluating the radial coefficients in Eq. (18) and carrying out the integrals of Eq. (20). Of course both steps are much lengthier than the corresponding steps in a zero-range calculation, even though the sum in Eq. (21) has exactly the same structure as in zero range.

In the special case that the form factor $f_{lm}(r_b,r_a)$ has the parity $\Pi=(-)^l$, under simultaneous inversion of $\mathbf{r}_b$ and $\mathbf{r}_a$, the comparison between zero range and finite range becomes especially close. Many important physical applications have this property, which we may call "normal" parity. For example, in deuteron stripping (with neglect of tensor forces and the deuteron D state),

$$f_{lm}(\mathbf{r}_p,\mathbf{r}_d)=\psi_{lm}^*(\mathbf{r}_n)D(r_{pn}),$$

and

$$\mathbf{r}_n=2\mathbf{r}_d-\mathbf{r}_p,\quad r_{pn}=2|\mathbf{r}_d-\mathbf{r}_p|.$$

The parity here is the parity of the bound-neutron wave function. With normal parity, Eq. (18) yields the selection rule

$$L_a+L_b+l=\text{even number}.$$

This same selection rule appears automatically in *all zero-range calculations*, because in these calculations Eq. (18) reduces to an integral over three spherical harmonics which have the same argument; such an integral is proportional to the factor $\langle L_bl,00|L_a,0\rangle$, which vanishes unless $L_a+L_b+l=$ even number.[10] Evidently for the case of normal parity the sum in Eq. (21) includes the same terms as in the zero-range case, and we can rewrite the amplitude (21) so as to display formally the geometrical factors which are used in the zero-range calculations.[2,9,11] The expressions become

$$\beta^{lm}=(2\pi^{1/2}/k_ak_b)\sum_{L_bL_a}\Gamma_{L_bL_a}{}^{lm}P_{L_b}{}^m(\theta)J_{L_bL_a}{}^l,\quad(22)$$

where

$$J_{L_bL_a}{}^l=(-)^{L_b}2\pi^{1/2}I_{L_bL_a}{}^l/(2L_b+1)^{1/2}\langle L_bl00|L_a0\rangle,\quad(23)$$

and the gamma coefficient

$$\Gamma_{L_bL_a}{}^{lm}=i^{L_a-L_b-l}(2L_b+1)[(L_b-m)!/(L_b+m)!]^{1/2}\times\langle L_bl00|L_a0\rangle\langle L_blm,-m|L_a0\rangle,\quad(24)$$

is identical to that used in the zero-range approximation.[9,11] In this form the only change required from current zero-range calculations is in the technique of evaluation of the radial integrals $J_{L_bL_a}{}^l$. The generalization to a calculation which includes spin-orbit coupling in the distorted waves is simple. The Γ are replaced by more complicated geometrical coefficients which depend also on s, j, m_a, m_b, J_a, and J_b (where $\mathbf{J}_a=\mathbf{L}_a+\mathbf{s}_a$, etc.), and which also appear in the corresponding zero-range calculation. At the same time the corresponding spin-orbit distorted waves have to be used. Otherwise the structure of the formalism is unchanged. (The formalism which has been coded includes the possibility of spin-orbit coupling for spin-$\frac{1}{2}$ or spin-1 particles.)

The most difficult step of the finite-range calculation lies in the evaluation of the "nonlocal" kernels $F_{lL_bL_a}(r_b,r_a)$ of Eq. (18). The subsequent double radial integral of Eq. (20) is reasonably convenient for numerical integration, once the nonlocal factors have been determined. These factors incorporate essentially all the physics which distinguishes different types of reactions, and different models for a particular reaction. It is worth noting that they do not depend upon the bombarding energy (provided V is energy independent) or upon the parameters which characterize the distorted waves. Hence, a particular set of kernel functions $F_{lL_bL_a}$ may be used in many different calculations. Since, by far, the greater part of the computing time goes into evaluating these functions, considerable savings may be made this way.

It must be expected that special methods will be developed to evaluate these nonlocal kernels in different cases. Later sections are devoted to discussions of some special methods. Evidently a well-designed numerical procedure should treat the calculation of the nonlocal kernels as a distinct unit, which can easily be changed. However, one trick used for evaluating the nonlocal kernels is general, and shall now be described.

For $l=0$ the integral of Eq. (18) is the expansion of a scalar. As a result three of the four angle integrations may be done immediately, and

$$F_{0LL}(r_b,r_a)=(-)^L2\pi(2L+1)^{1/2}\times\int_{-1}^{1}f_{00}(\mathbf{r}_b,\mathbf{r}_a)P_L(\mu)d\mu,\quad(25)$$

where $\mu=(\hat{r}_a\cdot\hat{r}_b)$ is the cosine of the angle between $\mathbf{r}_a$ and $\mathbf{r}_b$. We shall now show that for $l\neq0$ it is always possible to reduce Eq. (18) to a *finite* linear combination of integrals of the type of Eq. (25). In general, $f_{lm}(\mathbf{r}_b,\mathbf{r}_a)$ is a product of several factors, each of which may be nonscalar and contain a spherical harmonic of the polar angles of a vector $\mathbf{r}$ which is some linear combination of $\mathbf{r}_a$ and $\mathbf{r}_b$. Many reaction models are included in a form of f_{lm} which has three factors, at least one of which is scalar. A general treatment of such a form is given in the Appendix. The basic transformation used is one which converts the spherical harmonic $Y_l{}^m(\hat{r})$, where

$$\mathbf{r}=s\mathbf{r}_a+t\mathbf{r}_b,$$

[11] R. H. Bassel, G. R. Satchler, R. M. Drisko, and E. Rost, Phys. Rev. **128**, 2693 (1962).

into spherical harmonics in $\hat{r}_a$ and $\hat{r}_b$ separately, namely,[12]

$$r^l Y_l^m(\hat{r})=\sum_{\lambda\mu}[4\pi/(2\lambda+1)]^{1/2}\binom{2l+1}{2\lambda}^{1/2}(sr_a)^{l-\lambda}(tr_b)^\lambda \times\langle l-\lambda\lambda m-\mu\mu|lm\rangle Y_{l-\lambda}^{m-\mu}(\hat{r}_a)Y_\lambda^\mu(\hat{r}_b), \quad (26)$$

where λ runs from 0 to l, and

$$\binom{x}{y}=\frac{x!}{y!(x-y)!}$$

is the binomial coefficient. [The derivation of Eq. (26) may be based on the principle that a solid harmonic of a vector $\mathbf{r}$ is a homogeneous polynomial in the Cartesian coordinates of $\mathbf{r}$.] Because the values of l which are of most interest are usually rather small, the sum in Eq. (26) includes only a few terms, and leads to a very convenient reduction of Eq. (18) into integrals over the one variable μ. Illustrations of the application of this method are given in the next section.

The zero-range equivalent of a form factor must be so normalized as to yield the same results yielded by the exact form factor in the limit that the distorted waves have infinite wavelength. It therefore is

$$f_{lm}^{(\text{zero})}(\mathbf{r}_b,\mathbf{r}_a)=\delta(\mathbf{r}_b-(M_A/M_B)\mathbf{r}_a) \times\int f_{lm}((M_A/M_B)\mathbf{r}_a+\mathbf{s},\mathbf{r}_a)d\mathbf{s}. \quad (27)$$

The radial integrals $J_{L_bL_a}^{\ l}$ computed in finite range may be compared one-for-one with those computed from Eq. (27), to see in detail what are the finite-range modifications.

IV. STRIPPING

Here the projectile is assumed made up of the emitted particle b and another particle x which is captured by the target, so that the course of the reaction is

$$a+A\rightarrow(b+x)+A\rightarrow b+(x+A)\rightarrow b+B.$$

[For a (d,p) deuteron-stripping reaction, for example, we have $a=d$, $b=p$, $x=n$.] The interaction responsible for the transition is taken to be V_{bx}, usually assumed central- and spin-independent, so that $V_{bx}=V_{bx}(r_{bx})$. In deuteron stripping this interaction is V_{pn}. We also usually consider reactions in which b and x are initially in an s state of relative motion within a, and only give attention to more complicated kinds of relative motion for x and A within B. This is appropriate for deuteron stripping; stripping from non-S states is discussed in the next section. We therefore develop a scheme of analysis for the special case of S-state projectiles and central interactions, with the understanding that it is applicable for deuteron stripping, and that it is also applicable to many other stripping reactions by simple changes of numerical values of parameters.

From the discussion of the preceding sections it is evident that the form factor is

$$f_{lm}(\mathbf{r}_b,\mathbf{r}_a)=i^l\psi_{lm}^*(\mathbf{r}_{xA})D(r_{bx}), \quad (28)$$

where

$$\psi_{lm}(\mathbf{r})=i^l u_l(r)Y_l^m(\hat{r}) \quad (29)$$

is the normalized wave function for the bound state of the captured particle, that is, for the relative motion of A and x within B. The factor D was discussed in Sec. II. It is the product of the interaction V_{bx} times the internal wave function of a. In our special case D is a scalar,

$$D(r_{bx})=V_{bx}(r_{bx})\psi_a(r_{bx}).$$

In zero range, D becomes proportional to a δ function. We now rewrite the form factor in terms of the standard variables $\mathbf{r}_b$ and $\mathbf{r}_a$, the displacements of b and a from B and A, respectively. The transformation of variables is determined from the geometry, as shown in Fig. 1, giving

$$\mathbf{r}_{xA}=\alpha(\mathbf{r}_a-\gamma\mathbf{r}_b), \quad \mathbf{r}_{bx}=\alpha(\mathbf{r}_b-\delta\mathbf{r}_a), \quad (30)$$

where α, γ, δ are various ratios,

$$\alpha=aB[x(A+a)]^{-1}, \quad \gamma=(b/a), \quad \delta=(A/B),$$

of the masses of particles a, A, b, B, and x. It is the form factor of Eq. (28), using the variables of Eq. (30), which must now be introduced into the calculation of the angular integrals of Eq. (18).

Because D is a scalar it is already of the convenient type treated in Eq. (25). However, the transformation of solid harmonics of Eq. (26) must be used for the wave function. The wave function is first rewritten as

$$\psi_{lm}(\mathbf{r})=i^l w_l(r)[r^l Y_l^m(\hat{r})]. \quad (31)$$

The factor $w_l(r)=r^{-l}u_l(r)$ is well behaved at $r=0$, inasmuch as any eigenfunction u_l must be proportional to r^l near $r=0$. Then we may use the expansion (26) for the solid harmonic in brackets in Eq. (31). The remaining, scalar, part of f_{lm} is the product $w_l(r_{xA})D(r_{bx})$, and is a function of the scalar variables

$$r_{xA}=\alpha[r_a^2+\gamma^2r_b^2-2\gamma r_a r_b\mu]^{1/2}, \quad r_{bx}=\alpha[\delta^2r_a^2+r_b^2-2\delta r_a r_b\mu]^{1/2}, \quad (32)$$

where $\mu=(\hat{r}_a\cdot\hat{r}_b)$ is the cosine discussed in connection with Eq. (25). The integration over μ is equivalent to an expansion of this scalar part of f_{lm},

$$w_l(r_{xA})D(r_{bx})=\sum_{K=0}^{\infty}(K+\tfrac{1}{2})g_K(r_b,r_a)P_K(\mu), \quad (33)$$

where, by inversion,

$$g_K(r_b,r_a)=\int_{-1}^{1}d\mu w_l(r_{xA})D(r_{bx})P_K(\mu). \quad (34)$$

[12] This result has also been derived by M. Moshinsky, Nucl. Phys. **13**, 104 (1959), and by M. K. Banerjee (private communication).

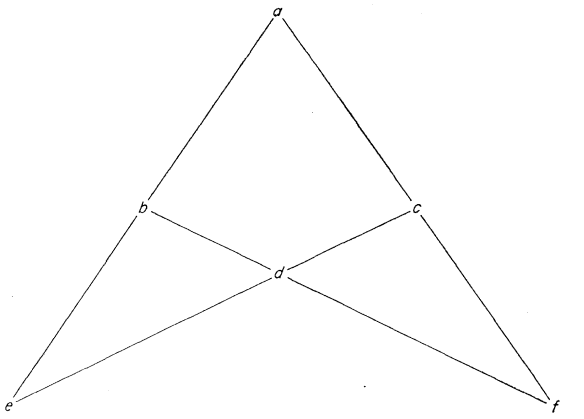

FIG. 2. Triangular inequalities for the Racah coefficient $W(abcd;ef)$. For this coefficient not to vanish, any three colinear quantities must be able to form a triangle.

Upon introducing these expansions into Eq. (18) the radial factors are finally found to be

$$F_{lL_bL_a}(r_b,r_a)=\alpha^l[\pi(2l+1)]^{1/2}\times\sum_{K\lambda}(r_a)^\lambda(-\gamma r_b)^{l-\lambda}(-)^K(2K+1)g_K(r_b,r_a)\times\binom{2l}{2\lambda}^{1/2}\langle\lambda K00|L_a0\rangle\langle l-\lambda K00|L_b0\rangle\times W(L_a\lambda L_b l-\lambda;Kl). \quad (35)$$

The W is a Racah coefficient which arises when we combine and integrate the pairs of spherical harmonics from Eqs. (33) and (31), and sum the associated Clebsch-Gordan coefficients over magnetic quantum numbers.[10] There are certain selection rules implied by Eq. (35) which restrict the numbers of terms contributing. For the Racah coefficient $W(abcd,ef)$ to be nonzero, all the connected triads in the pyramid shown in Fig. 2 must satisfy triangular inequalities, so that in the present case,

$$\begin{aligned}|L_b-l|\leq L_a\leq L_b+l,\\ 0\leq\lambda\leq l,\\ |L_a-\lambda|\leq K\leq L_a+\lambda,\\ |L_b-K|\leq l-\lambda\leq L_b+K.\end{aligned} \quad (36)$$

Further, parity conservation (exhibited in the Clebsch-Gordan coefficients) requires that only even values of (L_a+L_b+l), $(L_a+K+\lambda)$ and $(L_b+K+l-\lambda)$ enter. The first of these is the condition we called "normal parity," such that the finite-range calculation has the formal structure of a zero-range calculation with altered radial integrals, as in Eq. (22).

Under the above selection rules both the K and λ sums in Eq. (35) are finite, and are convenient to perform. The values of λ are small numbers, because l is a small number. The values of K are limited to a small range in the vicinity of L_a and L_b. Both these sums are consequences of the introduction of the finite-range interaction and collapse to closed form in the zero-range limit. We see this most easily by noting that in the zero-range limit

$$D(r_{bx})\rightarrow D_0\delta(\mathbf{r}_{bx}),$$

where D_0 is some appropriate constant, and that g_K then is easy to evaluate

$$g_K(\text{zero-range})=(D_0/2\pi)w_l(r_a)\delta(r_b-\delta r_a)/\alpha^3r_b^2. \quad (37)$$

This limiting form for g_K is independent of K, and, therefore, standard theorems about Racah coefficients[10] enable the K sum in Eq. (35) to be performed. The remaining sum on λ then is also easy, and finally,

$$F_{lL_bL_a}(\text{zero-range})=D_0r_b^{-2}\delta(r_b-\delta r_a)u_l(r_a)\times[(2L_a+1)(2L_b+1)/4\pi(2l+1)]^{1/2}\times\langle L_bL_a00|l0\rangle. \quad (38)$$

One of the more important points at which to identify finite-range modifications, therefore, is in the functional structure of the $g_K(r_b,r_a)$, the expansion coefficients of the scalar part of the form factor. With finite range it is clear that the g_K must drop off rapidly with K for the higher K values, merely because $P_K(\mu)$ in Eq. (34) is oscillatory, and it is clear that this effect must be most important at small radii. (As an extreme example of this, we note that in the limit that both w_l and D are constant, g_K vanishes unless $K=0$.) Finite-range modifications that are related to properties of the bound, final-state wave function $\psi_{lm}(\mathbf{r}_{xA})$ also show up in the K dependence of the g_K. The other source of finite-range modifications in the stripping amplitude is the oscillations of the distorted waves themselves. These oscillations affect the double radial integrals of Eq. (20) insofar as the g_K depart from δ-function form.

The calculational procedure used for stripping can now be summarized: (i) First Eq. (34) is applied, to determine the expansion coefficients $g_K(r_b,r_a)$. (ii) These coefficients are introduced into the sum, Eq. (35), and the radial factors $F_{lL_aL_b}(r_b,r_a)$ thereby determined. (iii) These radial factors are introduced into the double radial integral, Eq. (20), and the coefficients in Eq. (22) for the stripping amplitude thereby determined. (iv) Physical quantities of interest are evaluated from the amplitudes (22) as in standard zero-range calculations.

Step (i) in the above sequence, the determination of the $g_K(r_b,r_a)$, is the only step which is really troublesome. In our present procedure this step is performed by straightforward numerical integration of Eq. (34) for each value of K, and for all those pairs of values of r_b and r_a which lie in a band centered around the zero-range line, $r_b=\delta r_a$. This procedure is general. It may be used for applications in which D has an appreciable range, say, for reactions induced by heavy ions. If D actually has a fairly small range then only a narrow band of values of r_b, r_a need be used, and it

might be supposed that the method would simplify. Unfortunately this general method does not simplify, because adequate representation of a short-ranged D requires use of a finely spaced net in r_b, r_a. It is interesting, therefore, that Eq. (34) can be integrated analytically if D has Gaussian form, and if the wave function $w_l(r_{xA})$ can be treated as constant. Furthermore, the entire Eq. (34) can be integrated analytically if D has Gaussian form and if ψ_{lm} is an eigenfunction in a harmonic oscillator potential, and in this case the result of integration is a sum of modified Bessel functions of half-integer argument. This latter fact has disadvantages for practical calculations of deuteron stripping, because oscillator eigenfunctions are unphysical in the region of the nuclear surface and beyond, where usually the largest contributions to the reaction originate. Similar objections can be raised against other forms for w_l which can be expanded analytically. However, the analytic integration of Eq. (34) with a Gaussian D and oscillator w_l might be useful for checking the accuracy of a numerical procedure of integration.

Another useful check of the numerical procedure is obtained in the *plane-wave* limit. Although the use of this limit for the distorted waves $\chi_a^{(+)}$ and $\chi_b^{(-)}$ does not simplify the calculation of $g_K(r_b,r_a)$, it does yield a simple factored form for the over-all amplitude, as explained in Sec. II. The finite-range amplitude is then simply the zero-range amplitude multiplied by the Fourier transform of the range function D. Calculation using the numerically evaluated $g_K(r_b,r_a)$ must be able to reproduce this result.

If much calculation with fairly short-ranged functions D is to be performed, as, say, for deuteron stripping, then it might be profitable to shorten the calculation of Eq. (34) by expanding separately the two functions which enter. In terms of the two Legendre expansions

$$W_{lq}(r_b,r_a)=\int_{-1}^{1} d\mu w_l(r_{xA})P_q(\mu)\,, \tag{39}$$

$$D_p(r_b,r_a)=\int_{-1}^{1} d\mu D(r_{bx})P_p(\mu)\,, \tag{40}$$

we obtain for $g_K(r_b,r_a)$ itself the expansion

$$g_K(r_b,r_a)=\pi\sum_{p,q}[(2p+1)(2q+1)/(2K+1)] \times\langle pq00|K0\rangle^2 W_{lq}D_p. \tag{41}$$

Now special methods can be applied for the expansion of w_l and D separately, which cannot be applied for the expansion of their product. In particular, because D is not very precisely defined, physically, it is normally possible either to use a function D whose expansion can be computed analytically, or to choose a convenient analytic set of D_p as the definition of D. These coefficients would then be common to all stripping reactions initiated by a given type of projectile, and would therefore not need to be computed separately for each application. The expansion would then be very convenient if the coefficients W_{lq} should drop off sufficiently rapidly with q, so that only a few need be carried. [These coefficients certainly must be computed numerically, because any physically interesting Schrödinger eigenfunction $u_l(r_{xA})$ is only known numerically.] It is of interest, therefore, that if D has a short range, then the relevant W_{lq} do drop rapidly with q. For deuteron stripping probably only two or three of these coefficients need be carried, and these are available without much calculation.

The properties of the W_{lq} are seen by considering in detail the function $w_l(r_{xA})$,

$$w_l(r_{xA})=w_l(\alpha[r_a^2+\gamma^2 r_b^2-2\gamma r_a r_b\mu]^{1/2}) =w_l(\alpha[(r_a-\gamma r_b)^2+2\gamma r_a r_b(1-\mu)]^{1/2}). \tag{42}$$

A Legendre series in μ is a rearrangement of a power series in μ, or in $(1-\mu)$, and converges rapidly if either of these other series converges rapidly. Now a short range for the function D emphasizes the region $\mu\approx 1$ and $r_a\approx r_b$. Also, in deuteron stripping the parameter values in Eq. (42) are $\alpha\approx 2$, $\gamma\approx\frac{1}{2}$. Then for radii r_a, r_b which are large compared with the range of D it is clear from the argument in (42) that an expansion in $(1-\mu)$ has good convergence. For small r_a and r_b, good convergence follows for another reason—that in this region $u_l(r_{xA})$ is dominated by the centrifugal potential, and that, therefore, in this region $w_l(r_{xA})$ is approximately constant. A further way in which expansion of w_l is convenient is that the first two terms of the expansion involve the value and first two derivatives of w_l, which are readily obtained. Higher terms can in principle be obtained by using the Schrödinger equation as a recurrence relation for the power series expansion of w_l. As yet we have not tested this method numerically.

Finally, two alternative schemes which have been used for including finite-range effects in stripping reactions should be noted. In the first,[6] r_{xA} and r_{bx} are chosen as variables. Although this seems a natural choice when $D(r_{bx})$ is of short range, it involves a Taylor expansion of the *distorted waves*. This must be carried out anew in every calculation, whereas the g_K described above may be stored numerically and used in many calculations. A similar scheme[13] uses r_a and r_{bx} as variables, which involves expansion of one distorted wave, $\chi_b^{(-)}$, and of the bound-state wave function ψ_{lm}.

The second scheme[14] is closely related to the expansion proposed above for D and w_l separately, except that the whole calculation is carried out in momentum space. This then involves a Fourier expansion of the distorted waves, and again obscures the important and

[13] F. P. Gibson (private communication).
[14] D. Robson, Nucl. Phys. **42**, 592 (1963).

useful factorization expressed in, for example, Eqs. (3) and (15).

V. STRIPPING FROM NON-S STATES AND "HEAVY PARTICLE" STRIPPING

In the detailed discussion of the preceding section it was assumed that the transferred particle x was originally in an S state of motion relative to b when they formed the projectile a. We consider briefly here the changes involved when this is no longer true. Such a generalization is required, for example, if the effects of the deuteron D state are to be included in deuteron stripping, or if the concept of stripping is to be extended to other classes of reactions such as nucleon transfer in heavy-ion scattering. One such process, which has received considerable attention, is the so-called "heavy-particle" stripping (HPS). Suggested some years ago, this is supposed to resemble ordinary stripping except that the role of projectile and target nucleus are interchanged. The emitted particle b now originates in the target nucleus, and by analogy the interaction responsible for the transition is that which binds b to the rest of the target nucleus. There is still considerable discussion as to the physical importance of this reaction mode, and even as to its precise theoretical formulation. Nonetheless, bypassing the latter question, it is clear that any realistic study of the importance of HPS must take into account distortion effects.[14] Further, the zero-range approximation is hardly appropriate here (both the bound state of $b+x$ and the interaction V_{bx} now have "ranges" comparable to the size of the target nucleus). So it is of interest to discuss the application of the present techniques to HPS for the same simple 3-body model of stripping that was introduced in the previous section. The notation of the previous section immediately covers HPS if we interchange the interpretation of a and A as projectile and target, and write the HPS reaction as $a(A,b)B$.

The more complicated types of stripping reactions just mentioned all are special cases of the basic formalism of the present article. Only the procedure of calculating the radial kernels of Eq. (18) becomes more complicated. The principal new feature here just is that the internal angular momentum of nucleus a, formed from b and x, will generally be nonzero, and that as a result the form factor f_{lm} will include *two* factors which are nonscalar under rotations, *both* of which have to be expanded by application of Eq. (26). The general treatment of this type of form factor is given in the Appendix. Not only is the discussion in the Appendix a generalization of the stripping formalism, but also of the formalism for the exchange-knock-on process of our next section. There also the main text will emphasize a special case. The form of the general result for the radial kernels of Eq. (18) is seen in the Appendix to be very similar to that found in the last section, for example Eq. (35), except that the angular momentum algebra becomes more complicated. In particular the expansion of a nonscalar wave function ψ_a introduces an additional summation variable λ' with attendant angular momentum coupling factors.

VI. EXCHANGE-KNOCK-ON PROCESSES

Here the target nucleus A is assumed made up of the emitted particle b and another particle (or "core") c, which captures the incident projectile a, so that the course of the reaction is

$$a+A \rightarrow a+(b+c) \rightarrow b+(a+c) \rightarrow b+B.$$

The interaction responsible for the transition is taken to be V_{ba}, the entire physical interaction between particles a and b, and this we assume to be central and spin independent, as in Sec. IV. This V_{ba} presents some problems of principle, as it is partly responsible for the optical model interactions between a and A and b and B. However, the interest in the present paper concerns the kinematical structure of the integrals which arise, and therefore, to give the theory definiteness, we ignore these questions and just regard V_{ba} as a known, simple potential.

Several illustrations of exchange-knock-on processes may be mentioned. These are (p,n) reactions, the exchange terms in (p,p') reactions, and the knock-out term in (d,p) reactions. The last one of these examples concerns a term which might compete strongly with normal stripping if the target nucleus has a very loosely bound nucleon. In all these examples some antisymmetrization may be called for, and the knock-on amplitude may be only one term of a linear combination of interfering contributions from different reaction modes. In any case, it should be computed.

It is instructive to discuss a physical situation which presents a nontrivial and rather typical illustration of a knock-on process. We may imagine particles a and b to be spinless, and we may imagine nuclei A and B to each be described by a single parent state of the core c; in these states c has angular momentum J_c, and b and a have angular momenta l_b and l_a, respectively. Then the expansion (12) which defines the form factors f_{lm} is in general found to have several nonvanishing terms, arising from the coupling of l_a and l_b to various values of l. Each term is weighted by the coefficient A_l, whose l dependence is just that of the Racah coefficient $W(l_b,l_aJ_AJ_B;lJ_c)$. This coefficient controls the importance of the contributions which the various angular momentum transfers l make to the physical reaction amplitude. Of course if there is more than one parent state J_c, or if a, b have spins, the coefficients A_{lsj} would not be just Racah coefficients, but would be more complicated.[9] If in addition more than one pair of orbits l_a, l_b are involved, then each of the form factors f_{lm} which arise in Eq. (12) can be expressed as a sum of terms like the ones we are about to discuss. In any case we learn from the example that

in a knock-on process several terms in Eq. (12) are likely to be important. Of course it remains true that the different terms of (12) make incoherent contributions to the differential cross section, provided spin-orbit coupling can be neglected.

The form factor for our simple model is determined from Eqs. (13) and (14) to be

$$f_{lm}(\mathbf{r}_b,\mathbf{r}_a)=i^l\sum_{\mu_a\mu_b}\langle l_a l_b\mu_a,\ -\mu_b|lm\rangle(-)^{l_b-\mu_b}\times\psi_{l_a\mu_a}{}^*(\mathbf{r}_{ac})V_{ba}(r_{ba})\psi_{l_b\mu_b}(\mathbf{r}_{bc}),\quad(43)$$

where the bound-state wave functions are

$$\psi_{l\mu}(\mathbf{r})=i^l u_l(r)Y_l{}^\mu(\hat{r}).\quad(44)$$

From the geometry shown in Fig. 3, the variables in Eq. (43) are related by

$$\mathbf{r}_{ac}=\gamma'(\mathbf{r}_a+\alpha'\mathbf{r}_b),\quad \mathbf{r}_{bc}=\gamma'(\mathbf{r}_b+\beta'\mathbf{r}_a),\quad \mathbf{r}_{ba}=\eta\mathbf{r}_b-\zeta\mathbf{r}_a,\quad(45)$$

where the mass-ratio coefficients have the values

$$\gamma'=AB(AB-ab)^{-1},\quad \alpha'=(a/B),\quad \beta'=(b/A),$$
$$\eta=cB(AB-ab)^{-1},\quad \zeta=cA(AB-ab)^{-1}.$$

In the limit that the masses of the particles a, b are negligible compared with the masses of the nuclei A, B, it is seen that α' and β' become negligible, but that γ', η, ζ take on the value unity. An expansion technique which is useful when α' and β' are nonzero but small is described in the Appendix.

To compute the DWB amplitude from Eq. (43) we follow the standard procedure of Sec. III. It is necessary to compute the radial coefficients $F_{lL_aL_b}(r_b,r_a)$, and these are obtained by substituting Eq. (43) into Eq. (18). The transformation of Eq. (26) may be used to eliminate the complicated spherical harmonics $Y_{l_a}{}^{\mu_a}(\hat{r}_{ac})$ and $Y_{l_b}{}^{\mu_b}(\hat{r}_{bc})$ in favor of spherical harmonics of $\hat{r}_a$ and $\hat{r}_b$ separately. The calculations are straightforward, and are described in the Appendix. However, it is interesting to give attention to the case in which the projectile masses a, b are negligible compared with A, B. Then the coefficients α', β' go to zero and Eq. (26) need not be used. This case is interesting because it still includes the major effects of finite range on the knock-out process. We expand the scalar interaction V in spherical harmonics,

$$V=\sum_{K=0}^{\infty}(K+\tfrac{1}{2})v_K(r_b,r_a)P_K(\mu),$$
$$=2\pi\sum_{KQ}v_K(r_b,r_a)Y_K{}^{Q*}(\hat{r}_b)Y_K{}^Q(\hat{r}_a).\quad(46)$$

Substituting expansion (46) into Eq. (43), and this into Eq. (18), yields the radial kernels

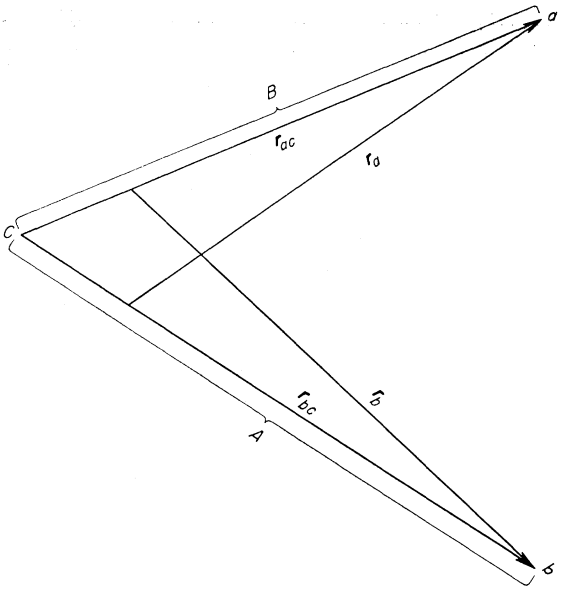

FIG. 3. The coordinate vectors for exchange-knock-on.

$$F_{lL_bL_a}(r_b,r_a)=\tfrac{1}{2}i^{l-l_a-l_b}[(2l_a+1)(2l_b+1)]^{1/2}u_{l_a}(r_a)u_{l_b}(r_b)\times\sum_K(-)^K(2K+1)\langle l_aK00|L_a0\rangle\langle l_bK00|L_b0\rangle\times W(L_al_aL_bl_b;Kl)v_K(r_b,r_a).\quad(47)$$

Eq. (47) is symmetrical under the interchange of the pairs L_a, l_a and L_b, l_b. The angular momentum coupling in Eq. (47) is controlled by the Racah coefficient. According to Fig. 2, certain triads must satisfy triangular inequalities, so that

$$l_b+l_a\geq l\geq|l_b-l_a|,$$
$$L_b+L_a\geq l\geq|L_b-L_a|,$$
$$L_b+l_b\geq K\geq|L_b-l_b|,$$
$$L_a+l_a\geq K\geq|L_a-l_a|.\quad(48)$$

The Clebsch-Gordan coefficients enforce the parity rules, that only even values of (l_a+K+L_a) and (l_b+K+L_b) shall enter.

Equation (47) has much the same structure as Eq. (35) in the case of stripping. The bound-state quantum numbers l_a, l_b usually are small numbers, and this restricts the summation. But now the subsequent summations, in computing the amplitude, do not take on an equivalent "zero-range form" because the parity is not "normal." The parity is $(-)^{l_a+l_b}$, rather than $(-)^l$, and odd values of $(l+l_a+l_b)$ and $(l+L_a+L_b)$ are allowed. Once again the summation over K collapses in the zero-range limit (because in this limit v_K becomes independent of K and may be removed from the sum), and only "normal" parity terms survive.

In order to anticipate some of the finite-range effects which may appear in the knock-on amplitude it is once again interesting to examine the plane-wave limit. If the no-recoil form of Eq. (43) is substituted into Eq. (15), and plane waves are used, then we obtain

$$(2l+1)^{1/2}\beta^{lm}=\sum_{\mu_a\mu_b}\langle l_a l_b\mu_a,\ -\mu_b|lm\rangle(-)^{l_b-\mu_b}\times\int d\mathbf{r}_a\int d\mathbf{r}_b V(r_{ba})\psi_{l_a\mu_a}{}^*(\mathbf{r}_a)\psi_{l_b\mu_b}(\mathbf{r}_b)\times\exp i[\mathbf{k}_a\cdot\mathbf{r}_a-\mathbf{k}_b\cdot\mathbf{r}_b]. \quad (49)$$

This equation is more complicated than the corresponding equation for stripping, and does not automatically factor. However, an approximate factoring does appear if we can assume that the range of V is small enough so that the bound-particle wave functions may be evaluated at the average position variable, $\mathbf{r}=\frac{1}{2}(\mathbf{r}_b+\mathbf{r}_a)$. Then

$$(2l+1)^{1/2}\beta^{lm}\approx\sum\langle l_a l_b\mu_a,\ -\mu_b|lm\rangle(-)^{l_b-\mu_b}\times\int d\mathbf{r}\psi_{l_a\mu_a}{}^*(\mathbf{r})\psi_{l_b\mu_b}(\mathbf{r})\exp[i\mathbf{r}\cdot(\mathbf{k}_a-\mathbf{k}_b)]\times\int d\mathbf{r}_{ba}V(r_{ba})\exp[-\tfrac{1}{2}i\mathbf{r}_{ba}\cdot(\mathbf{k}_a+\mathbf{k}_b)]. \quad (50)$$

The two integrals in Eq. (50) involve momentum transfer vectors which have very different structures. Thus, the first integral tends to be large at forward scattering angles, where $\mathbf{k}_a-\mathbf{k}_b$ is small. Then at such forward angles the momentum $\mathbf{k}_a+\mathbf{k}_b$ is large, so that strong finite-range effects are suggested by the second integral. The two integrals tend to reverse their roles at large scattering angles. In either event, strong finite-range effects are expected. A similar discussion is given in a recent article by Rodberg,[15] and the conclusions are confirmed by explicit calculation of proton scattering from carbon,[16] and of the (n,p) reaction on silicon.[17]

VII. SUMMARY AND DISCUSSION

Exact numerical calculations with finite-range interactions are much more difficult than with zero-range interactions. However, we have shown that the finite-range calculations can be arranged so as to be practical. The methods which are described in this article have been built into a code for the IBM 7090 computer, and preliminary results have been obtained. The code and detailed numerical results will be described in another article.

In the qualitative discussion it has been seen that finite-range corrections for deuteron stripping are not expected to be drastic, except in possibly suppressing contributions from the nuclear interior. However, the plane-wave estimates did suggest that other reaction processes may be much more sensitive to finite-range effects.[15]

For deuteron stripping, the present indications from the exact numerical calculations are that the finite-range effects only lead to a partial suppression of contributions from the nuclear interior. The shape of the differential cross-section curve tends to be altered rather little, and it is still often necessary to use radial cutoffs on the stripping integrals in order to obtain agreement with experiment.[6] When cutoffs are not used, there is a noticeable over-all reduction of the magnitude of the cross section, but only to the degree expected from the suppression of interior contributions. When a cutoff close to the nuclear surface is employed, finite-range effects are very small. However, more striking effects have been obtained for other processes, such as (p,α), as expected from the larger momentum transfers involved.

Both in Eq. (9) and in Eq. (49) it is seen that *in plane-wave approximation* the finite-range correction factor is of the nature of a Fourier transform of a two-body interaction. This transform can fall off rapidly if the relevant momentum difference should become large. For example, if the two-body interaction should be of Gaussian shape then the rate at which the Fourier transform drops with momentum may become very rapid, and toward large scattering angles the cross section may drop by several orders of magnitude. It is especially interesting that the numerical calculations show that such drastic finite-range effects tend to disappear when distorted waves are used. With distorted waves there is no tendency for the introduction of the finite-range interaction to force the cross section down to exceptionally small values.

The influence of distorted waves in a finite-range calculation was previously discussed in Sec. II, and it was pointed out that distortion tends to introduce into the wave functions higher momenta than are present in the plane waves. This tends to enhance finite-range effects. However, more generally, distortion *spreads* the momentum distribution of the wave functions, and not only high momenta but also low momenta are introduced. This allows effects from low-momentum differences to dominate at scattering angles where the plane-wave theory would lead one to expect strong effects caused by high-momentum differences. For this reason the high-momentum parts of the Fourier transform of the two-body interaction are not very important. They are small. How small they are does not matter, because enough of the large, low-momentum parts of the transform enter the calculation to dominate the results. Two interactions which agree at low momenta tend to give indistinguishable

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cross sections. (This last fact is of practical use, because it means that the convenient Gaussian interaction may be employed without noticeable error, so long as distortions are appreciable.)

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APPENDIX

Many models for the "effective nuclear interaction" of Eqs. (2) and (12) can be written as a sum of terms, each of which has the structure

$$\phi_{l_1\mu_1}{}^*(\mathbf{r}_1)\phi_{l_2\mu_2}(\mathbf{r}_2)u_3(r_3)\,. \tag{51}$$

In the expression (51), u_3 is a scalar, while in general the

$$\phi_{l_i\mu_i}(\mathbf{r})=u_i(r_i)Y_{l_i}{}^{\mu_i}(\hat{r}_i) \tag{52}$$

are not, but carry angular momentum l_i. In general the arguments have the form

$$\mathbf{r}_1=s_1\mathbf{r}_a+t_1\mathbf{r}_b\,,\quad \mathbf{r}_2=s_2\mathbf{r}_a+t_2\mathbf{r}_b\,,$$
$$r_3=|s_3\mathbf{r}_a+t_3\mathbf{r}_b|\,. \tag{53}$$

Several examples of the form (51) have been discussed already. Stripping from S-state projectiles (Sec. IV) is of this form with $l_2=0$ and $\mathbf{r}_2=\mathbf{r}_3=\mathbf{r}_{bx}$. The relations corresponding to Eq. (53) are given in Eq. (30). The extension to non-S-state stripping (Sec. V) consists of $l_2\neq 0$. In the knock-on process (Sec. VI), $\phi_{l_1\mu_1}$ and $\phi_{l_2\mu_2}$ correspond to the orbits into which a is captured and from which b is ejected, respectively, while u_3 corresponds to the interaction V_{ba}. The relations between $\mathbf{r}_i$, $\mathbf{r}_a$, and $\mathbf{r}_b$ are then given by Eq. (45). In the limit of no recoil, $t_1=s_2=0$; we shall return later to the interesting case that t_1/s_1 and s_2/t_2 are small.

The terms (51) are always required in the covariant combinations which behave like $Y_l{}^{m*}$ under coordinate rotations, as in Eqs. (12) and (14),

$$h_{lm}=\sum_{\mu_1\mu_2}\langle l_1l_2\mu_1,\,-\mu_2|lm\rangle(-)^{\mu_2}\phi_{l_1\mu_1}{}^*\phi_{l_1\mu_2}u_3\,. \tag{54}$$

Each such term corresponds to transfer of angular momentum l, m, and parity change, $(-)^{l_1+l_2}$. The h_{lm} in Eq. (54) should also be labeled with l_1 and l_2, and possibly other quantum numbers, but these will be omitted for simplicity.

The general form (54) has some advantages for numerical computation. The properties of a specific reaction model are contained in the radial functions $u_i(r_i)$, and the coefficients s_i, t_i, but the angular momentum algebra now to be described is common to all such calculations. Thus, a flexible computer code may carry options for the u_i and have the s_i, t_i as input parameters. Exceptions occur when special cases such as $l_2=0$ (Sec. IV) or neglect of recoil (Sec. VI) are considered. The algebra then simplifies sufficiently to justify separate calculational procedures.

With the form factor (54) we need the radial coefficients in the bipolar expansion corresponding to Eq. (17),

$$h_{lm}(\mathbf{r}_b,\mathbf{r}_a)$$
$$=\sum_{L_aL_bM}H_{lL_bL_a}(r_b,r_a)Y_{L_b}{}^{M*}(\hat{r}_b)Y_{L_a}{}^{m-M*}(\hat{r}_a)$$
$$\times\langle L_bL_aMm-M|lm\rangle\,. \tag{55}$$

This expansion is easily carried out using Eq. (26) and the generalization of Eqs. (33) and (34), namely

$$g_K(r_b,r_a)=\int_{-1}^{1}d\mu P_K(\mu)w_1(r_1)w_2(r_2)u_3(r_3)\,, \tag{56}$$

where $\mu=\hat{r}_b\cdot\hat{r}_a$ and $w_i=u_i/r_i{}^{l_i}$. The result is

$$H_{lL_bL_a}(r_b,r_a)$$
$$=\tfrac{1}{2}\sum_{\lambda_1\lambda_2K}(s_1r_a)^{l_1-\lambda_1}(t_1r_b)^{\lambda_1}(s_2r_a)^{\lambda_2}(t_2r_b)^{l_2-\lambda_2}g_K(r_b,r_a)$$
$$\times\sum_{\Lambda_a\Lambda_b}(-)^{L_b-l_2+\lambda_2-\lambda_1}(2l_1+1)(2l_2+1)(2K+1)$$
$$\times[(2\Lambda_a+1)(2\Lambda_b+1)]^{\frac{1}{2}}\times\langle\Lambda_aK00|L_a0\rangle$$
$$\times\langle\Lambda_bK00|L_b0\rangle\langle l_1-\lambda_1\lambda_200|\Lambda_a0\rangle\langle l_2-\lambda_2\lambda_100|\Lambda_b0\rangle$$
$$\times W(L_aL_b\Lambda_a\Lambda_b;lK)\binom{2l_1}{2\lambda_1}^{\frac{1}{2}}\binom{2l_2}{2\lambda_2}^{\frac{1}{2}}\begin{bmatrix} l & l_1 & l_2 \\ \Lambda_a & l_1-\lambda_1 & \lambda_2 \\ \Lambda_b & \lambda_1 & l_2-\lambda_2 \end{bmatrix}. \tag{57}$$

In the expression (57), the sums are limited by the following triads which must obey triangular inequalities: $(L_a\Lambda_aK)$, $(L_b\Lambda_bK)$, $(\Lambda_al_1-\lambda_1,\lambda_2)$, $(\Lambda_bl_2-\lambda_2,\lambda_1)$, (lL_aL_b), (ll_1l_2), $(l\Lambda_a\Lambda_b)$, of which the first four must also have an even sum. The last factor in Eq. (57) is the 9-j symbol[10]; numerically this is computed as an expansion over a product of three Racah coefficients. Of these three, two are very simple to compute because each contains one argument which is the arithmetic sum of two others.[10]

The simple examples discussed in the main text are limiting cases of Eq. (57). Putting $l_2=0$ (so that $\lambda_2=0$, $\Lambda_a=l_1-\lambda_1$, $\Lambda_b=\lambda_1$, and $l_1=l$), Eq. (57) reduces to the form of Eq. (35). The no-recoil limit of Sec. VI is also of interest. The sum over λ_1, λ_2 in Eq. (57) arises from the expansion of the spherical harmonics in $\phi_{l_1\mu_1}$ and $\phi_{l_2\mu_2}$, in the form factor (54). With the no-recoil model for knock-on reactions, this expansion is no longer necessary because $\mathbf{r}_1=\mathbf{r}_a$, $\mathbf{r}_2=\mathbf{r}_b$. Then $\lambda_1=\lambda_2=0$ only, so that $\Lambda_a=l_1$, $\Lambda_b=l_2$ and Eq. (57) reduces to the form of Eq. (47).

It is also of interest to consider an approximate scheme for use when t_1/s_1 and s_2/t_2 are very small, so that only the values 0 and 1 need be considered for λ_1 and λ_2. This corresponds, for example, to taking recoil

into account to first order in the knock-on reaction. It may also be applied to "heavy particle" stripping if the target is sufficiently massive. Explicit expressions for this case could be obtained by substitution in Eq. (57), but a more convenient and consistent approach is to make a Taylor expansion of the $\phi_{l\mu}$,

$$\phi_{l_1\mu_1}(\mathbf{r}_1)=\phi_{l_1\mu_1}(s_1\mathbf{r}_a)+t_1\mathbf{r}_b\cdot\nabla\phi_{l_1\mu_1}(s_1\mathbf{r}_a)+\cdots, \quad (58)$$

(where ∇ is the gradie t with respect to $s_1\mathbf{r}_a$), and similarly for $\phi_{l_2\mu_2}$. Then to first order in t_1 and s_2 we may write

$$H_{lL_bL_a}=H_{lL_bL_a}{}^{(0)}+H_{lL_bL_a}{}^{(1)}, \quad (59)$$

where the zero-order term

$$H_{lL_bL_a}{}^{(0)}(r_b,r_a)$$
$$=\tfrac{1}{2}\sum_K g_K(r_b,r_a)u_1(s_1r_a)u_2(t_2r_b)(-)^K(2K+1)$$
$$\times\langle l_1K00\,|\,L_a0\rangle\langle l_2K00\,|\,L_b0\rangle[(2l_1+1)(2l_2+1)]^{\frac{1}{2}}$$
$$\times W(L_aL_bl_1l_2;lK), \quad (60)$$

has the same form as the no-recoil expression, Eq. (47). The first-order correction is

$$H_{lL_bL_a}{}^{(1)}(r_b,r_a)$$
$$=\tfrac{1}{2}\sum_{\nu\eta K}(2\nu+1)(2\eta+1)(2K+1)g_K(r_b,r_a)(-)^{L_b+l_1+l}$$
$$\times[l_1r_bu_2(t_2r_b)d_\nu(s_1r_a)+s_2r_au_1(s_1r_a)d_\eta(t_2r_b)]$$
$$\times\langle\nu K00\,|\,L_a0\rangle\langle\eta K00\,|\,L_b0\rangle\langle\nu100\,|\,l_10\rangle\langle\eta100\,|\,l_20\rangle$$
$$\times W(l_1l_2\nu\eta;l1)W(L_aL_b\nu\eta;lK). \quad (61)$$

Since the expansion (58) has been used for the functions ϕ, the g_K in Eqs. (60) and (61) arises from the expansion of $u_3(r_3)$ alone,

$$g_K(r_b,r_a)=\int_{-1}^{1}d\mu P_K(\mu)u_3(r_3). \quad (62)$$

When u_3 represents a two-body interaction potential, it will often be possible to use forms such as a Gaussian for which analytic expressions are available for the g_K. The radial parts of expression (61) also involve the derivatives of u_1 and u_2 through the combinations d_ν and d_η, where

$$d_{l_i+1}(x)=[(d/dx)-(l_i/x)]u_i(x),$$
$$d_{l_i-1}(x)=[(d/dx)+(l_i+1/x)]u_i(x). \quad (63)$$

The summations in Eq. (61) are quite limited. The only values of ν and η allowed are $\nu=l_1\pm1$, $\eta=l_2\pm1$. The first Racah coefficient in Eq. (61) therefore has simple explicit forms, as also do the second two Clebsch-Gordan coefficients. The remainder of Eq. (61) has the same structure as the no-recoil term, Eq. (60).

Finally we consider the special case of this approximation where $\mathbf{r}_3=\mathbf{r}_2$. This would be the case for heavy-particle stripping. It is then unnecessary to carry out the expansion (62); $u_3(r_2)$ may be combined with $u_2(r_2)$,

$$u_3(r_2)u_2(r_2)=\tilde{u}_2(r_2).$$

Since this is equivalent to replacing u_3 by unity in the integral (62), this leads to $g_K=2$ if $K=0$, but zero otherwise; that is, the K sum disappears from Eqs. (60) and (61). Explicitly the results are quite simple,

$$H_{lL_bL_a}{}^{(0)}=\delta_{L_al_1}\delta_{L_bl_2}u_1(s_1r_a)\tilde{u}_2(t_2r_b)(-)^{L_a+L_b-l}$$

and

$$H_{lL_bL_a}{}^{(1)}$$
$$=-[(2l_1+1)(2l_2+1)]^{1/2}W(L_aL_bl_1l_2;l1)$$
$$\times\langle l_1100\,|\,L_a0\rangle\langle l_2100\,|\,L_b0\rangle$$
$$\times[t_1r_b\tilde{u}_2(t_2r_b)d_\nu(s_1r_a)$$
$$+s_2r_au_1(s_1r_a)d_\eta(t_2r_b)],$$

where now d_η acts on $\tilde{u}_2$. Only the values $L_a=l_1\pm1$, $L_b=l_2\pm1$ enter into $H^{(1)}$.